# Linear Algebra Review

by Anthony Bonner

- Symmetric matrices
- Eigenvectors / eigenvalues
- Spectral decomposition
- Positive semi-definite matrices

Lemma 1: For real symmetric matrices, the eigenvectors of distinct eigenvalues are orthogonal.

Proof:

let $x$ & $y$ be eigenvectors of symmetric matrix $A$, & let $\alpha$ & $\beta$ be their eigen values

$\therefore \quad Ax = \alpha x$

& $\quad Ay = \beta y$

$\therefore \quad y^T A x = \alpha y^T x \qquad *$

& $\quad x^T A y = \beta x^T y$

$\therefore \beta y^T x$

$= (\beta x^T y)^T$

$= (x^T A y)^T$

$= y^T A^T x$

$= y^T A x$ since $A = A^T$

$= \alpha y^T x$ by $*$

$\therefore \beta y^T x = \alpha y^T x$

$\therefore (\beta - \alpha) y^T x = 0$

If the eigenvalues are distinct, then $\beta \neq \alpha$

$\therefore y^T x = 0$

ie $y \perp x$

qed.

<u>Corollary 2:</u> an $n \times n$ symmetric matrix has at most $n$ distinct eigenvalues, since it can have at most $n$ orthogonal eigenvectors.

<u>Claim</u> (hard).

An $n \times n$ symmetrix matrix has <u>exactly</u> $n$ orthogonal eigenvectors.

<u>Note</u>: If $x$ is an eigenvector of matrix $A$, then so is $ax$, for all $a \in \mathbb{R}$, $a \neq 0$, since

$$Ax = \lambda x$$

$$\text{iff} \quad a(Ax) = a(\lambda x)$$

$$\text{iff} \quad A(ax) = \lambda(ax)$$

So, we can choose the eigenvectors of a matrix to have any length we want.

**Theorem 3:** Let $A$ be a real symmetric matrix. Then

$$A = ODO^T$$

where

- $O = (o_1 \cdots o_n)$

$$\left.\begin{array}{ll} - \; A\sigma_i = \lambda_i \sigma_i & \\ - \; \|\sigma_i\| = 1 & \\ - \; \sigma_i \perp \sigma_j & i \neq j \end{array}\right\} \text{the } \sigma_i \text{ are orthonormal eigenvectors of } A.$$

- $D$ is a diagonal matrix whose $i^{th}$ diagonal entry is $\lambda_i$, the $i^{th}$ eigenvalue of $A$.

**Proof**

Let $o_1 \cdots o_n$ be the $n$ orthogonal eigenvectors of $A$.

$\therefore \quad A o_i = \lambda_i o_i$

& $\quad o_i \perp o_j \quad$ for $i \neq j$

Also, we can assume that $\|o_i\| = 1$.

Let $\quad O = (o_1 \cdots o_n)$

$\therefore \; O^T O = I \quad$ since $\; (O^T O)_{ij} = o_i^T o_j$

$\therefore \; O^{-1} = O^T$

(ie, $O$ is an orthogonal matrix).

$$A\sigma_i = \lambda_i \sigma_i \qquad \text{For} \quad i = 1 \cdots n$$

$$\therefore (A\sigma_1, \cdots, A\sigma_n) = (\lambda_1 \sigma_1, \cdots, \lambda_n \sigma_n)$$

$$\therefore A(\sigma_1 \cdots \sigma_n) = (\sigma_1 \cdots \sigma_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

ie, $AO = OD$

$\therefore \cancel{AO}$

$$\therefore A = ODO^{-1} = ODO^T$$

qed.

Corollary 4:

The determinant of a real symmetric matrix is the product of its eigenvalues.

Proof.

$$
\begin{aligned}
\operatorname{Det} A &= \det (O D O^{T}) \\
&= (\det O) \cdot (\det D) \cdot (\det O^{T}) \\
&= (\det O) \cdot (\det O^{T}) \cdot (\det D) \\
&= \det (O O^{T}) \cdot \det D \\
&= \det I \cdot \det D \\
&= \det D \\
&= \lambda_1 \cdots \lambda_n
\end{aligned}
$$

since $D$ is diagonal.

qed.

Lemma 5:

A diagonal matrix, D, with non-negative diagonal entries is positive semi-definite, i.e., $z^T D z \geq 0 \quad \forall z \in \mathbb{R}^n$.

Proof. Let $\lambda_1 \cdots \lambda_n$ be the diagonal entries of D, & let $z = (z_1 \cdots z_n)^T$

$$\therefore z^T D z = \sum_{i=1}^{n} z_i^2 \lambda_i$$
$$\geq 0 \quad \text{since } \lambda_i \geq 0.$$

Corollary 6:

If $A$ is a real symmetric matrix, then it is positive semi-definite iff all its eigenvalues are non-negative.

Proof $A = ODO^T$

if direction:

If all the eigenvalues of $A$ are non-negative, then the diagonal entries of $D$ are non-negative, so $D$ is positive semi-definite.

$$\therefore\ z^T A z = z^T O D O^T z$$
$$= (O^T z) D (O^T z)$$
$$\geq 0$$

$\therefore$ $A$ is positive semi-def.

<u>only if direction:</u>

Suppose $A$ is positive semi-def.

Let $e_i = (0 \cdots 0\,1\,0 \cdots 0)^T$

$\quad\quad\quad\quad\quad\quad\uparrow$ ith position

Let $z_i = O e_i$

$\therefore$ $e_i = O^T z_i$

Let $\lambda_i$ be the $i^{\underline{th}}$ eigenvalue of $A$.

$\therefore \lambda_i$ is the $i$th diagonal element of $D$.

$$\therefore \lambda_i = e_i^T D e_i$$
$$= (O^T z_i)^T D (O^T z_i)$$
$$= z_i^T O D O^T z_i$$
$$= z_i^T A z_i$$
$$\geq 0$$ since $A$ is positive semi-definite.

qed.

Corollary 7.

If $A$ is a symmetric, positive semi-definite matrix, then $A = BB^T$ for some matrix $B$.

Proof. $A = ODO^T$ where the diagonal entries of $D$ are non-negative.

ie, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where $\lambda_i \geq 0$

Let $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$

note. $\sqrt{D} \cdot \sqrt{D} = D$

$$\therefore\ A = O\sqrt{D}\sqrt{D}\,O^T$$

$$= O\sqrt{D}\,(\sqrt{D})^T O^T$$

$$= (O\sqrt{D})(O\sqrt{D})^T$$

$$= BB^T$$

where $B = O\sqrt{D}$

<u>qed</u>.

Corollary 8.

$A$ is a symmetric, positive semi-def matrix iff $A = BB^T$ for some matrix $B$.

Proof.

only-if direction.

This is corollary 7.

if direction

If $A = BB^T$

then $A^T = (BB^T)^T$

$= (B^T)^T B^T$

$= BB^T = A$

$\therefore$ $A$ is symmetric.

Also,

$$
\begin{aligned}
z^T A z &= z^T B B^T z \\
&= (B^T z)^T (B^T z) \\
&= \| B^T z \|^2 \\
&\geq 0
\end{aligned}
$$

$\therefore$ $A$ is pos. semi-def.

qed.

## Corollary 9.

If $A$ is a symmetric, positive semi-definite matrix, then $A = B^2$ for some ~~post~~ symmetric, positive semi-definite matrix, $B$.

(Note: we often say $B = \sqrt{A}$).

**Proof:** $A = ODO^T$

where $O^T O = I$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ is diagonal}$$

& $\lambda_i \geq 0$ (by corollary 6)

Let $C = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$

$\cancel{\therefore C = C^T}$ $\therefore C^2 = D$

$$\therefore A = ODO^T$$
$$= OC^2O^T$$
$$= OC(O^TO)CO^T \qquad \text{since } O^TO = I$$
$$= (OCO^T)(OCO^T)$$
$$= B^2 \qquad \text{where} \qquad B = OCO^T$$

note that $B$ is symmetric + positive semi-definite (since the diagonal entries of $C$ are non-negative).

qed.

# Covariance Matrices



Let $v = (v_1 \dots v_m)^T$ where each $v_i$ is a real-valued random variable with mean 0, ie, $E(v_i) = 0$.

Definition. The covariance matrix of $v$, ~~is~~ denoted $\text{cov}(v)$, is the matrix $B$ where $B_{ij} = E(v_i v_j)$.

Thm 10: $\text{Cov}(v)$ is sym and positive semi-definite.

proof. Symmetry is easy:

$$B_{ij} = E(v_i v_j) = E(v_j v_i) = B_{ji}$$

$$\therefore B = B^T$$

To show positive semi-definiteness,
let $z = (z_1 \cdots z_m)^T \in \mathbb{R}^m$.

$$
\begin{aligned}
\therefore\ z^T B z &= \sum_{ij} z_i B_{ij} z_j \\
&= \sum_{ij} z_i E(v_i v_j) z_j \\
&= E\left(\sum_{ij} z_i v_i v_j z_j\right) \\
&= E\left[\left(\sum_i z_i v_i\right) \cdot \left(\sum_j z_j v_j\right)\right] \\
&= E\left[\left(\sum_i z_i v_i\right)^2\right] \\
&\geq 0
\end{aligned}
$$

$\therefore$ B is pos. semi-def.

qed.

# Pointwise Multiplication

**Lemma 11:**

If $A$ & $B$ are symmetric, pos. semi-def matrices, then so is $C$, where $C_{ij} = A_{ij} B_{ij}$.

**Proof:** Let $v$ & $w$ be independent random vectors, where

$$v = (v_1 \cdots v_m)^T \sim \mathcal{N}(0, A)$$
$$w = (w_1 \cdots w_m)^T \sim \mathcal{N}(0, B)$$

ie, $v$ & $w$ are normally distributed with mean $0$ and

$$\left.\begin{aligned} A &= \operatorname{cov}(v) \qquad \text{ie, } A_{ij} = E(v_i v_j) \\ B &= \operatorname{cov}(w) \qquad \text{ie, } B_{ij} = E(v_i v_j) \end{aligned}\right\} *$$

Let $X = (X_1 \cdots X_m)^T$ where $X_i = V_i W_i$

$\therefore E(X_i) = E(V_i W_i)$

$= E(V_i) \cdot E(W_i)$ by independence

$= 0 \cdot 0 = 0$

$\therefore [\mathrm{cov}(X)]_{ij}$

$= E(X_i X_j)$ by definition

$= E[(V_i W_i)(V_j W_j)]$

$= E[(V_i V_j) \cdot (W_i W_j)]$

$= E(V_i V_j) \; E(W_i W_j)$ by independence

$= A_{ij} B_{ij}$ by $*$

$= C_{ij}$ by definition

$\therefore C = \mathrm{cov}(X)$.

$\therefore C$ is sym. & pos. semi-def, by Thm 10.

q.e.d.